Speed-Up of the Strongly Implicit Procedure with Application to Subsonic/Transonic Potential Flows

Rachel Gordon*
Technion-Israel Institute of Technology, Haifa, Israel
and
Rimon Arieli†
Rafael, Haifa, Israel

Abstract

A modified version of the Strongly Implicit Procedure (SIP) for solving partial differential equations is presented. The capabilities of the modified scheme are demonstrated and analyzed for the case of two-dimensional subsonic or transonic potential flow over airfoils. The improved technique accelerates the convergence rate of the SIP considerably. For the problem studied here, it was found to reduce the number of iterations required for convergence by 50–85%, when compared to the SIP. The modified technique is weakly sensitive to the choice of the parameter sequence α of the SIP, and its optimal relaxation parameters are almost independent of the grid size, grid type, or type of the equations to be solved.

Introduction

Stone's Strongly Implicit Procedure (SIP)[1] is a fully implicit iterative method for solving large sets of linear equations that arise in the approximate solution of partial differential equations. The method was used for solving various problems.[1-3] The method was later extended by Rubin and Kosla[4] to the solution of block algebraic equations that arise in the solution of a set of strongly coupled partial differential equations.

In this study an improved technique is presented for accelerating the convergence rate of the SIP. The technique is based on the use of overrelaxation parameters that are combined with alternating sequences of solutions. The new technique has been tested on the solution of steady, subsonic/transonic potential flow over a two-dimensional airfoil. Both techniques have been used before. Overrelaxation is a well-known technique that is used with many numerical techniques. Alternating solutions were tried before with the SIP[1-3] and were found to improve the convergence rate in some cases. In all the cases that we examined, overrelaxation by itself caused the SIP to diverge, and alternating solutions did not improve the convergence rate. However, by combining the two techniques, we achieve a significant improvement of the convergence rate of the SIP.

Physical Test Case and Mathematical Formulation

Two-dimensional, steady, subsonic/transonic potential flow is governed by the conservation laws of mass and energy and the isentropic gas assumption.[2] The equations may be recast in a strongly conservative form into a general computational coordinate system (ξ,ζ) yielding the following set of equations:

$$\left(\frac{\rho U}{J}\right)_\xi + \left(\frac{\rho W}{J}\right)_\zeta = 0 \tag{1}$$

$$\rho = \left[1 + \frac{1}{2}(\gamma - 1)M_\infty^2(1 - (U\phi_\xi + W\phi_\zeta))\right]^{1/(\gamma-1)} \tag{2}$$

where γ is the specific heats ratio, M_∞ is the freestream Mach number and

$$J = \xi_x\zeta_z - \xi_z\zeta_x; \qquad U = A_1\phi_\xi + A_3\phi_\zeta; \qquad W = A_3\phi_\xi + A_6\phi_\zeta$$

$$A_1 = \xi_x^2 + \xi_z^2; \qquad A_6 = \zeta_x^2 + \zeta_z^2; \qquad A_3 = \xi_x\zeta_x + \xi_z\zeta_z$$

An implicit finite difference approximation for Eq. (1) is written resulting in a linear set of equations. The equations are then recast into a "correction" form: $\phi^{n+1} = \phi^n + \delta\phi^{n+1}$ (n denotes the iteration number) yielding the following linear set of equations:

$$B\cdot\delta\phi_{i,k-1}^{n+1} + D\cdot\delta\phi_{i-1,k}^{n+1} + E\cdot\delta\phi_{i,k}^{n+1} + F\cdot\delta\phi_{i+1,k}^{n+1} + H\cdot\delta\phi_{i,k+1}^{n+1} = R_{i,k}(\phi^n) \tag{3}$$

The detailed derivation and definitions of the coefficients can be found in Ref. 5. Equation (3) can be written in a matrix notation as

$$[M]\cdot[\delta\phi^{n+1}] = [R(\phi^n)] \tag{4}$$

where the coefficient matrix $[M]$ is a five-diagonal matrix for the two-dimensional case under consideration.

The boundary conditions along the outer boundaries are

$$\rho_{ff} = 1; \qquad \phi_{ff} = \phi_\infty + \frac{\Gamma}{2\pi}\tan^{-1}\left[\sqrt{1-M_\infty^2}\cdot\left(\frac{z}{x}\right)\right] \tag{5a}$$

where $\phi_\infty = x\cos\alpha_A + z\sin\alpha_A$, α_A is the mean angle of attack of the airfoil, and the circulation Γ is given by the potential jump at the trailing edge. The Γ is unknown a priori and is calculated by an iterative procedure. On the airfoil the boundary conditions are

$$W = 0, \text{ or alternatively } \left(\frac{\rho W}{J}\right) = 0 \tag{5b}$$

Two slightly different, body-fitted, sheared C-type grids described in Refs. 6 and 7 were used in this study.

Strongly Implicit Procedure and Accelerated Strongly Implicit Procedure

A complete description of Stone's SIP method can be found in Ref. 1; therefore we shall describe it here only briefly. The SIP alters the matrix $[M]$ to obtain a matrix $[M']$ that can be factored into the product of a lower triangular matrix $[L]$ and an upper triangular matrix $[U]$, each containing only three nonzero diagonals: $[L]\cdot[U] = [M] \cong [M']$. Simple recursion equations are then used to solve the modified set of equations. The computational molecule of $[M']$ is built from the five

Received Feb. 13, 1989; revision received Aug. 25, 1989.

*Koret Fellow, Department of Aeronautical Engineering, the Technion; currently Visiting Assistant Professor, Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843.

†Research Associate. Member AIAA.

points of $[M]$ and an additional pair of asymmetric grid points. To reduce the effect of the additional terms, the method incorporates a relaxation parameter α, which is allowed to vary between 0 and 1. A different algorithm can be obtained by incorporating the opposite pair of asymmetric grid points for each grid node point. The recursion equations of this algorithm can be found in Ref. 3. This later algorithm will be called here the "backward solution" to distinguish it from the former "forward solution." These two algorithms are independent of each other and can be used by themselves or can be coupled into an alternating sequence of odd and even iteration numbers. As for the computational effort, the time required for a given number of iterations of the SIP when used with the forward solution alone is the same as with the backward solution alone, and it is also the same as with the alternating solutions (alternating sweeps are counted as two iterations). The amount of storage required by the above three variants is also the same.

Our studies show that the use of an alternating sequence of solutions together with overrelaxation accelerates the convergence rate of the SIP considerably. This modified algorithm will be called the ASIP (Accelerated Strongly Implicit Procedure). For the flow problem studied here, overrelaxation is used in updating the velocity potential ϕ and the circulation Γ in the following manner:

$$\phi^{n+1,*} = (1-\omega_\phi)\phi^n + \omega_\phi \phi^{n+1} \equiv \phi^n + \omega_\phi \delta\phi^{n+1}$$
$$\Gamma^{n+1,*} = (1-\omega_\Gamma)\Gamma^n + \omega_\Gamma \Gamma^{n+1} \equiv \Gamma^n + \omega_\Gamma \delta\Gamma^{n+1} \qquad (6)$$

where ω_ϕ and ω_Γ are relaxation parameters for ϕ and Γ respectively, and the superscript * denotes the updated value.

Our studies show that the use of overrelaxation parameters greater than 1 with just one type of solution, the forward solution or the backward solution, causes divergence of the iterative procedure. Furthermore, for $\omega_\phi = \omega_\Gamma = 1$ the convergence speed of the forward solution and of the ASIP are about the same.

Typical Results

Results were obtained for two-dimensional subsonic or transonic flow over a NACA0012 airfoil at various angles of attack. The parameter α of the SIP was taken as a cyclic set of M elements given by $\alpha_m = 1 - 0.001^{m/M}$, $m = 0,1,\ldots,M-1$.

The convergence criterion in the present calculations was

$$\text{RMAX} = \frac{\max_{i,k} \cdot |R_{i,k}^{n+1}|}{\max_{i,k} \cdot |R_{i,k}^{1}|} \le C_r \text{ and } \left|\frac{(\Gamma^n - \Gamma^{n-M})}{\Gamma^n}\right| \le C_r \qquad (7)$$

where C_r is a prescribed convergence parameter, taken in the present calculations to be 10^{-4}.

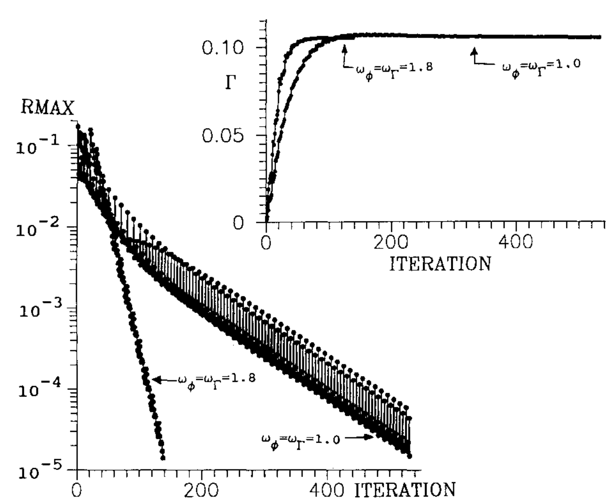

Fig. 1 Convergence histories of RMAX and Γ of the ASIP.

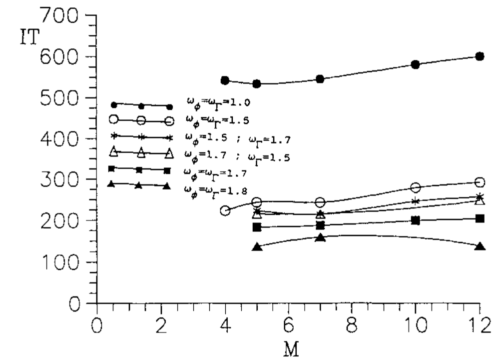

Fig. 2 The number of iterations required for convergence vs M.

Figure 1 presents typical results of comparison of the convergence histories of the maximum residual RMAX, and Γ, of the ASIP for $\omega_\phi = \omega_\Gamma = 1$ and $\omega_\phi = \omega_\Gamma = 1.7$. The results were obtained for $M_\infty = 0.75$ and $\alpha_A = 1$ deg with a 140×35 C-type grid of Ref. 6. At these flow conditions, a shock wave occurs. Figure 2 shows typical results of the number of iterations required by the ASIP for convergence as a function of the number of elements M for various relaxation parameters. The results were obtained for a subcritical flow of $M_\infty = 0.75$ and $\alpha_A = 2$ deg, with a 140×35 C = type grid of Ref. 6.

Conclusions

An improved technique for accelerating the SIP method has been presented. The advantages of the technique are as follows. a) It does not depend strongly on the nature of the coefficient matrix of the equations to be solved. b) It is not sensitive to the choice of the relaxation parameter sequence α. c) There are optimal overrelaxation parameters, which reduce the number of iterations required for convergence to a minimum. These parameters are almost independent of the grid size, grid type, type of the equations to be solved, or size of the α sequence for $M \ge 5$. These parameters are about $\omega_\phi = \omega_\Gamma = 1.7$. For $M \le 4$, both the SIP and the ASIP diverge for most of the cases that we investigated. d) It reduces significantly the number of iterations required for convergence. For the test cases studied here, a 50-85% reduction was achieved. e) The computer storage required by the procedure is the same as that of the SIP.

The technique is expected to be an important tool in improving the SIP method when used for solving large sets of linear equations that arise in the approximate solution of sets of partial differential equations.

References

[1]Stone, H. L., "Iterative Solution of Implicit Approximations of Multi-Dimensional Partial Differential Equations," *SIAM Journal of Numerical Analysis,* Vol. 5, pp. 530–558, Sept. 1968.

[2]Malone, J. B., and Sankar, N. L., "Numerical Simulation of 2-D Unsteady Transonic Flows Using the Full-Potential Equation," AIAA Paper 83-0233, Jan. 1983.

[3]Pepper, D. W., and Harris, S. D., "Fully Implicit Algorithm for Solving Partial Differential Equations," *Journal of Fluids Engineering*, Vol. 99, No. 4, Dec. 1977, pp. 781–783.

[4]Rubin, S. G., and Kosla, K. P., "Navier-Stokes Calculations with a Coupled Strongly Implicit Method," *Journal of Computers and Fluids,* Vol. 9, No. 2, 1981, pp. 163–180.

[5]Arad, E., Gordon, R., and Arieli, R., "An Improved Technique for the Solution of Inviscid Transonic 3D Flows." *Proceedings of the 29th Annual Conference of the Israel Society of Aviation and Astronautics*, The Technion —I.I.T., Haifa, Israel, Feb. 1987, pp. 1–8.

[6]Woan, C. J., and Bonner, E., "An Investigation of Gridding Effect on the Accuracy of FLOW5117MM and FLOW57MG Euler Wing Solution," AIAA Paper 85-4090, Oct. 1985.

[7]Sorenson, R. L., "A Computer Program to Generate Two-Dimensional Grids about Airfoils and other Shapes by the use of the Poisson's Equation," NASA TM-81198, May 1980.